Information collecting situations and bi-monotonic allocation schemes

Brânzei, R.; Tijs, S.H.; Timmer, J.B.

Published in:
Mathematical Methods of Operations Research

Publication date:
2001

Math Meth Oper Res (2001) 54:303–313

Mathematical Methods
of Operations Research

Information collecting situations and bi-monotonic allocation schemes

Rodica Brânzei[1], Stef Tijs[2], Judith Timmer[2,3]

[1] Faculty of Computer Science, "Al.I. Cuza" University, 16 Berthelot St., 6600 Iaşi, Romania.
[2] CentER and Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.
[3] Current address: Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands (e-mail: j.b.timmer@math.utwente.nl)

Manuscript received: March 2001/Final version received: June 2001

Abstract. This paper studies information collecting (IC) situations with the help of cooperative game theory. Relations are established between IC situations and IC games on one hand and information sharing (IS) situations and IS games on the other hand. Further, it is shown that IC games are convex combinations of so-called local games. Properties such as k-convexity and k-concavity are possessed by an IC game if all related local games have the respective properties. Special attention is paid to the classes of k-symmetric IC games and k-concave IC games. This last class turns out to consist of total big boss games. For the class of total big boss games a new solution concept is introduced: bi-monotonic allocation schemes. This solution takes the power of the big boss into account.

JEL Classification Number: C71, D89.

Key words: cooperative games, information collecting, big boss games, bi-monotonic allocation scheme

1 Introduction

In Brânzei, Tijs and Timmer (2001) information collecting (IC) situations and corresponding IC games are introduced. They model situations where an action taker in an uncertain situation can improve upon his action choices by gathering information from agents who are more informed about the situation. The problem of sharing the gains, when cooperating with informants, is tackled by constructing the corresponding IC game and considering solutions developed for such games.

This paper continues the study of IC situations and related games. We start by comparing IC situations and information sharing (IS) situations, as introduced by Slikker, Norde, and Tijs (2000). They model situations where all the agents are action takers in an uncertain situation and where they can gain by sharing their information about the situation. It turns out that any IC situation is an IS situation and conversely, any IS situation is related to n IC situations, where n is the number of agents.

After this, we introduce local games corresponding to IC situations and show that an IC game is a suitable convex combination of its local games. Properties as convexity and k-concavity of the local games are inherited by the corresponding IC games. For k-symmetric games a geometric method is described to discover whether the game is k-concave, convex or neither.

Upon studying k-concave IC games we arrive at a characterization of such games with the help of big boss games, as extensively studied in Muto, Nakayama, Potters and Tijs (1988). This characterization says that an IC game is k-concave if and only if it is a total big boss game. Inspired by the special structure of the core of total big boss games, we introduce bi-monotonic allocation schemes (bi-mas) for IC games. These allocation schemes take the veto power of the single action taker into account, namely, if the action taker cooperates with a larger group of informants then the part of the total reward allocated to him will not decrease while the parts allocated to the informants will not increase. Hence, in larger coalitions the action taker is better off while the informants are worse off. It is shown that any core element of a total big boss game can be extended to a bi-mas.

The organization of this paper is as follows. Section 2 introduces IC situations and IS situations and corresponding games. Relations between both situations are investigated. In Section 3 we introduce local games corresponding to an IC situation and show that their properties are inherited by the IC game. The geometric method for k-symmetric IC games is described in Section 4. Finally, Section 5 is devoted to total big boss games and bi-monotonic allocation schemes. Here a characterization of k-concave IC games is given and it is shown that all core elements of a total big boss game can be extended to a bi-mas.

2 Information collecting and information sharing

In an information collecting (IC) situation a single agent, the decision-maker, has to choose an action a from some action set. The reward resulting from this choice also depends upon the true state of the world, which is not known by the decision-maker. Unfortunately the decision-maker has to act before the true state obtains. He has some information about the true state and this information is described by a partition of the finite set Ω of all possible states of the world. An element of such a partition is called an event and if the true state obtains then the decision-maker knows which event happens, that is, he knows which element of his partition contains the true state. Next to the decision-maker there are also other agents that have information about the uncertainty. These agents can be consulted by the decision-maker.

An IC situation with decision-maker k is denoted by a tuple

$$\langle N, k, (\Omega, \mu), \{\mathscr{I}_i\}_{i \in N}, A_k, r_k \rangle$$

where $N = \{1, 2, \ldots, n\}$ is the set of all agents including the decision-maker k.

Ω is the finite set of all states of the world and μ is a positive probability distribution on Ω, which means that $\mu(\omega) > 0$ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} \mu(\omega) = 1$. The information of agent $i \in N$ is represented by the partition $\mathscr{I}_i$ of Ω. The set A_k is the finite set of actions available for the decision-maker and $r_k(\omega, a_k)$ is his reward if ω is the true state and if action a_k is chosen. We assume that

$$\sum_{I \in \mathscr{I}_k} \max_{a_k \in A_k} \sum_{\omega \in I} r_k(\omega, a_k)\mu(\omega) \geq 0,$$

which means that if the decision-maker works on his own then he can achieve a nonnegative expected reward. Collecting information from $S \backslash \{k\} \subset N$ results in the expected payoff

$$\sum_{I \in \mathscr{I}_S} \max_{a_k \in A_k} \sum_{\omega \in I} r_k(\omega, a_k)\mu(\omega). \tag{1}$$

Here $\mathscr{I}_S = \{\bigcap_{i \in S} I_i \mid I_i \in \mathscr{I}_i, \bigcap_{i \in S} I_i \neq \emptyset\}$ is the partition of Ω that describes the total information of all the agents in S.

The cooperative game (N, v_k) related to such an IC situation with decision-maker k is defined by the player set N and the function $v_k(S)$ that assigns $v_k(S) = 0$ to coalition S if $k \notin S$ and $v_k(S)$ is given by (1) if $k \in S$. A cooperative game that corresponds to an IC situation is called an IC game. The examples below illustrate these concepts.

Example 2.1. Consider the IC situation

$$\langle N, k, (\Omega, \mu), \{\mathscr{I}_i\}_{i \in N}, A_k, r_k \rangle$$

where $N = \{1, 2, 3, 4\}$, $k = 4$, $\Omega = A_4 = \{0, 1\}^3$, $\mu(\omega) = 1/8$ for each $\omega \in \Omega$, $\mathscr{I}_4 = \{\Omega\}$, $\mathscr{I}_i = \{\{x \in \Omega \mid x_i = 0\}, \{x \in \Omega \mid x_i = 1\}\}$ for $i \in \{1, 2, 3\}$, $r_4(\omega, a) = 160$ if $\omega = a$, $r_4(\omega, a) = 0$ otherwise.

One may interpret this as follows. A monster is hidden in one of the eight rooms, 000, 100, 010, 001, 110, 101, 011, and 111, of an apartment. Agent 4 has to find the monster but he may only choose one room. If the monster is in that room he obtains a reward of 160, otherwise there is no reward. He gets the possibility to consult the agents 1, 2 and 3. Agent $i \in \{1, 2, 3\}$ knows the i-th coordinate of the room number. The corresponding IC game (N, v_4) is given by $N = \{1, 2, 3, 4\}$, $v_4(S) = 0$ for all $S \subset N$ with $4 \notin S$, $v_4(\{4\}) = 20$, $v_4(\{4, i\}) = 40$ for $i \in \{1, 2, 3\}$, $v_4(\{4, i, j\}) = 80$ if $i, j \in \{1, 2, 3\}$ and $i \neq j$, and $v_4(N) = 160$.

Example 2.2. Take the situation as in example 2.1 and modify it in such a way that player 4 can choose three rooms to look for the monster. This leads to the IC situation

$$\langle N, 4, (\Omega, \mu), \{\mathscr{I}_i\}_{i \in N}, A_4', r_4' \rangle$$

where $A_4' = \{K \subset \Omega \,\big|\, |K| = 3\}$, $r_4'(\omega, K) = 160$ if $\omega \in K$, $r_4'(\omega, K) = 0$ otherwise. This leads to the IC game (N, v_4') with $v_4'(S) = 0$ if $4 \notin S$, $v_4'(\{4\}) = 60$, $v_4'(\{i, 4\}) = 120$ for $i \in \{1, 2, 3\}$ and $v_4'(S) = 160$ if $|S| \geq 3$ and $4 \in S$.

Note that the games (N, v_4) and (N, v_4') in the examples 2.1 and 2.2 are monotonic, $v(S) \leq v(T)$ if $S \subset T$, with 4 as veto player, $v(S) = 0$ if $4 \notin S$. These properties follow from the definition of an IC game.

Slikker, Norde and Tijs (2000) introduce information sharing (IS) situations and corresponding games. In an IS situation any agent has to choose an action and his resulting reward depends upon his choice and the true state of the world. The agents may consult each other to obtain more information about the true state.

An IS situation is denoted by a tuple

$$\langle N, (\Omega, \mu), \{\mathscr{I}_i, A_i, r_i\}_{i \in N} \rangle$$

where N, Ω, μ, and $\mathscr{I}_i$ have the same meaning as for IC situations. Now there is for *each* agent $i \in N$ a nonempty finite action set A_i and a reward function $r_i : \Omega \times A_i \to \mathbb{R}$. Agent i has to choose an action $a_i \in A_i$ and obtains the reward $r_i(\omega, a_i)$ depending on the chosen action and the true state ω.

We assume that

$$\sum_{I \in \mathscr{I}_i} \max_{a_i \in A_i} \sum_{\omega \in I} r_i(\omega, a_i)\mu(\omega) \geq 0$$

for all $i \in N$, on his own any agent can obtain a nonnegative expected reward. If a group S of agents decides to cooperate and share their information, then the total expected reward for this group is given by

$$\sum_{i \in S} \sum_{I \in \mathscr{I}_S} \max_{a_i \in A_i} \sum_{\omega \in I} r_i(\omega, a_i)\mu(\omega). \tag{2}$$

The corresponding IS game (N, v) is the game where $v(S)$ is given by (2) for all coalitions S and $v(\emptyset) = 0$.

What can we say about relations between IC situations and IS situations? First, suppose we have an IC situation $\langle N, k, (\Omega, \mu), \{\mathscr{I}_i\}_{i \in N}, A_k^*, r_k^* \rangle$ leading to the IC game (N, v_k). Then this game is also the IS game corresponding to the IS situation

$$\langle N, (\Omega, \mu), \{\mathscr{I}_i, A_i, r_i\}_{i \in N} \rangle$$

where $A_k = A_k^*$, A_i is an arbitrary nonempty finite set for $i \neq k$, $r_k = r_k^*$ and $r_i = 0$ if $i \in N \backslash \{k\}$. So, an IC situation with k as action taker can be transformed into an IS situation, where all action takers except k have a trivial reward function, the zero-function.

Next, we try the other way around. Suppose we have an IS situation

$$\langle N, (\Omega, \mu), \{\mathscr{I}_i, A_i, r_i\}_{i \in N} \rangle$$

with corresponding IS game (N, v). Consider the IC situations

$$\langle N, k, (\Omega, \mu), \{\mathscr{I}_i\}_{i \in N}, A_k, r_k \rangle$$

for all $k \in N$ with corresponding IC games (N, v_k), $k \in N$. According to (1), (2) and k being a veto player in (N, v_k), $v = \sum_{i \in N} v_i$. Hence, an IS situation can be decomposed in $|N|$ IC situations and the corresponding IS game is the sum of the IC games. In Slikker et al. (2000) it is shown that an IS game is a game with a population monotonic allocation scheme (pmas) (cf. Sprumont (1990)). In fact, the scheme $[a_{iS}]_{i\in S, S\in 2^N\setminus\{\emptyset\}}$ defined by $a_{iS}=0$ if $i\notin S$, and $a_{iS}=v_i(S)$ otherwise, is a pmas for the IS game (N,v) because

$$\sum_{i\in S} a_{iS} = \sum_{i\in S} v_i(S) = v(S)$$

as shown above and

$$a_{iS} \le a_{iT} \quad \text{for all } i\in S\subset T$$

because (N,v_i) is a monotonic game.

Example 2.3. Consider the IS situation $\langle N,(\Omega,\mu),\{\mathscr{I}_i,A_i,r_i\}_{i\in N}\rangle$ with $N=\{1,2\}$, $A_1=A_2=\Omega=\{00,01,10,11\}$, $\mu(00)=1/10$, $\mu(01)=\mu(10)=\mu(11)=3/10$, $\mathscr{I}_1=\{\{00,01\},\{10,11\}\}$ and $\mathscr{I}_2=\{\{00,10\},\{01,11\}\}$. The reward functions are

$$r_1(a,\omega)=20(|a_1-\omega_1|+|a_2-\omega_2|)$$

$$r_2(a,\omega)=40-10(|a_1-\omega_1|+|a_2-\omega_2|)$$

for all $a=(a_1,a_2)\in A_2$, $\omega\in\Omega$. One can think of the following situation. In an apartment with four rooms, 00, 01, 10 and 11, a monster is hidden. The probability that the monster is in room 00 is 1/10 and the probability that the monster is in one of the other rooms is 3/10. The agents 1 and 2 have to visit a room. Agent 1 likes a room as far away from the monster as possible, while agent 2 wants to be as close to the monster as possible. Agent 1 knows whether the monster is at the north side $\{00,01\}$ of the apartment or not, and agent 2 knows whether the monster is at the west side $\{00,10\}$ or not.

The corresponding IS game (N,v) is the sum of the two IC games (N,v_1) and (N,v_2), where $N=\{1,2\}$

$$v_1(\{1\})=32,\quad v_1(\{2\})=0,\quad v_1(\{1,2\})=40,$$

$$v_2(\{1\})=0,\quad v_2(\{2\})=36,\quad v_2(\{1,2\})=40.$$

The table below

S	$\{1\}$	$\{2\}$	N
a_{1S}	32	–	40
a_{2S}	–	36	40

contains the scheme $[a_{iS}]_{i\in S,S\in 2^N\setminus\{\emptyset\}}$ that is a pmas for the IS game (N,v).

3 Local IC games

In the previous Section we introduced IC games which are cooperative games corresponding to IC situations. In this Section we introduce so-called local IC games, which are also related to IC situations.

Let $\langle N, k, (\Omega, \mu), \{\mathcal{I}_i\}_{i\in N}, A_k, r_k\rangle$ be an IC situation. Given the fact that ω is the true state, a group S of agents knows that the event $I_S(\omega)$ happens, which is that element of $\mathcal{I}_S$ containing ω. Consequently, each $\omega' \in I_S(\omega)$ occurs with conditional probability

$$\mu(\omega')/\mu(I_S(\omega)).$$

The *local* IC game $(N, v_{k,\omega})$ assigns to each coalition S, $k \in S$,

$$v_{k,\omega}(S) = \max_{a_k \in A_k} \sum_{\omega' \in I_S(\omega)} r_k(a_k, \omega')\mu(\omega')/\mu(I_S(\omega)),$$

the expected reward if one knows that ω is the true state, and $v_{k,\omega}(S) = 0$ otherwise. Notice that $v_{k,\omega}(S) = v_{k,\omega'}(S)$ for all $\omega' \in I_S(\omega)$ and that, according to (1), for coalitions S with $k \in S$

$$\begin{aligned} v_k(S) &= \sum_{I\in\mathcal{I}_S} \max_{a_k \in A_k} \sum_{\omega' \in I} r_k(a_k, \omega')\mu(\omega') \\ &= \sum_{I\in\mathcal{I}_S} \mu(I) v_{k,\tilde{\omega}}(S) = \sum_{I\in\mathcal{I}_S}\sum_{\omega\in I} \mu(\omega)v_{k,\omega}(S) = \sum_{\omega\in\Omega}\mu(\omega)v_{k,\omega}(S) \end{aligned}$$

for some $\tilde{\omega} \in I$. Also, $v_k(S) = 0 = \sum_{\omega\in\Omega}\mu(\omega)v_{k,\omega}(S)$ if $k \notin S$. So, we have proved

Proposition 3.1. *If (N, v_k) is an IC game and $\{(N, v_{k,\omega})\}_{\omega\in\Omega}$ are the corresponding local games, then $v_k = \sum_{\omega\in\Omega}\mu(\omega)v_{k,\omega}$.*

The following proposition shows that local games can be of use in discovering special properties of an IC game. Interesting properties of IC games with k as action taker are k-concavity and k-convexity. Define $M_i(T, v_k) = v_k(T) - v_k(T\backslash\{i\})$, the marginal contribution of player i to coalition $T\backslash\{i\}$. An IC game (N, v_k) is called *k-concave* if

$$M_i(S, v_k) \geq M_i(T, v_k) \tag{3}$$

for all $i \in N\backslash\{k\}$ and $S, T \subset N$ with $\{i, k\} \subset S \subset T$. If

$$M_i(S, v_k) \leq M_i(T, v_k) \tag{4}$$

for $\{i, k\} \subset S \subset T$, $i \neq k$, then the game is called *k-convex*. Note that k-convex games are convex games (Shapley (1971)), which have many interesting properties. A thorough study of k-concave IC games will be given in Sect. 5. For the moment we are interested in relations between k-concave (convex) local IC games and their related IC game.

Proposition 3.2. *If all the local IC games $(N, v_{k,\omega})$ of an IC situation are k-concave (convex), then the corresponding IC game (N, v_k) is k-concave (convex).*

Proof. We only prove that (N, v_k) is k-concave, given that all local IC games are k-concave. It follows from proposition 3.1 that for all $i \neq k$, S and T with $\{i, k\} \subset S \subset T$

$$M_i(S, v_k) - M_i(T, v_k) = \sum_{\omega \in \Omega} (M_i(S, v_{k,\omega}) - M_i(T, v_{k,\omega}))\mu(\omega) \geq 0$$

where the inequality follows from (3) and $\mu(\omega) > 0$. □

This proposition is demonstrated in the following example.

Example 3.3. Consider the IC situation $\langle N, k, (\Omega, \mu), \{\mathcal{I}_i\}_{i \in N}, A_k, r_k \rangle$ where $N = \{1, 2, 3\}$, $k = 3$, $A_3 = \{n, g_a, g_b, g_c, g_d\}$, $\Omega = \{a, b, c, d\}$ and $\mu(\omega) = 1/4$ for all $\omega \in \Omega$. Further $\mathcal{I}_1 = \{\{a, b\}, \{c, d\}\}$, $\mathcal{I}_2 = \{\{a, b, c\}, \{d\}\}$, $\mathcal{I}_3 = \{\Omega\}$, $r_k(\omega, n) = 0$ for all $\omega \in \Omega$, $r_k(\omega, g_\omega) = 40$ and $r_k(\omega, g_x) = -60$ if $x \neq \omega$.

One can think of a situation where a treasure with a value of 40 dollars is hidden in one of the places a, b, c or d with equal probabilities. Agent 3 can guess where the treasure is, and if his guess is correct he obtains the treasure. If his guess is wrong he has to pay 60 dollars. Another action for agent 3 is not to guess (action n) with resulting reward 0. It only makes sense to guess if agent 3 knows where the treasure is. This implies that the local IC games are defined by $v_{3,a} = v_{3,b} = 0$, $v_{3,c}(\{1, 2, 3\}) = 40$ and $v_{3,c}(S) = 0$ otherwise, $v_{3,d}(\{2, 3\}) = v_{3,d}(\{1, 2, 3\}) = 40$ and $v_{3,d}(S) = 0$ otherwise. These local games are convex, so the IC game v_3 is also convex by proposition 3.2. According to proposition 3.1 $v_3 = (v_{3,a} + v_{3,b} + v_{3,c} + v_{3,d})/4$ and so, $v_3(\{2, 3\}) = 10$, $v_3(\{1, 2, 3\}) = 20$ and $v_3(S) = 0$ otherwise.

4 k-Symmetric IC games

For the special class of k-symmetric IC games we can discover whether the game is convex, k-concave or none of them by looking at the graph of the so-called detection function, as the proposition below shows.

An IC game (N, v_k) is called *k-symmetric* if

$$v_k(S) = v_k(T) \quad \text{for all } S, T \text{ with } k \in S, k \in T \text{ and } |S| = |T|.$$

Corresponding to such a k-symmetric game (N, v_k) we construct the detection function $B : [1, n] \to \mathbb{R}$ as follows. Denote by b_t the value $v_k(S)$ of a coalition S with $|S| = t$ and $k \in S$. Then B is defined by

$$B(x) = (x - t)b_{t+1} + (t + 1 - x)b_t \quad \text{for } x \in [t, t + 1]$$

where $t \in \{1, 2, \ldots, n - 1\}$. Notice that the graph of B coincides with the broken line in $\mathbb{R}^2$ obtained by connecting for $t \in \{1, 2, \ldots, n - 1\}$ the points (t, b_t) and $(t + 1, b_{t+1})$ with a line segment.

Proposition 4.1. *Let (N, v_k) be a k-symmetric IC game and let B be the corresponding detection function. Then*

(i) (N, v_k) *is a convex game if and only if* B *is a convex function.*
(ii) (N, v_k) *is a k-concave game if and only if* B *is a concave function.*
(iii) (N, v_k) *satisfies*

$$v_k(N) - v_k(S) \geq \sum_{i \in N \setminus S} M_i(N, v_k) \tag{5}$$

for all S *with* $k \in S$ *if and only if the graph of* B *does not lie above the line in* $\mathbb{R}^2$ *through the points* $(n-1, b_{n-1})$ *and* (n, b_n).

The proof of the proposition is left to the reader.

Example 4.2. First, consider again example 2.1. This leads to a 4-symmetric IC game with a convex detection function, so the game is convex.

Next, consider example 2.2. There we have a 4-symmetric IC game with a concave detection function, and so the game is 4-concave.

Knowing that the IC game (N, v_k) is a convex game discloses a lot of properties of the solutions of the game (cf. Shapley (1971)). For example, the Shapley value (cf. Shapley (1953)) lies in the core of the game. Also it is interesting to know whether (N, v_k) is k-concave as the following Section shows.

5 Total big boss games and bi-monotonic allocation schemes

In this Section we pay attention to IC games (N, v_k) that also satisfy the k-concavity condition (3). The k-concavity condition says that for a player i his marginal contribution to a smaller coalition containing k is (weakly) larger than his marginal contribution to a larger coalition also containing k. The first theorem below shows that an IC game (N, v_k) is k-concave if and only if it is a total big boss games with big boss k, which we define now. A game (N, v) is a *big boss game* with big boss k (cf. Muto et al. (1988), Tijs (1990)) if the game is monotonic, (5) is satisfied and player k is a veto player.

Let us call a monotonic game (N, v) with veto player k a *total big boss game* with k as big boss if the game itself and all subgames (T, v), $k \in T$, are big boss games. Stated otherwise, a monotonic game (N, v) with veto player k is a total big boss game with big boss k if and only if

$$v(T) - v(S) \geq \sum_{i \in T \setminus S} M_i(T, v) \tag{6}$$

for all S, T with $k \in S \subset T$.

Theorem 5.1. *Let* (N, v_k) *be an IC game. Then* (N, v_k) *is a total big boss game with big boss k if and only if it is k-concave.*

Proof. Let (N, v_k) be an IC game. Assume first that it is k-concave. Let $k \in S \subset T$. Suppose $T \setminus S = \{i_1, i_2, \ldots, i_h\}$. Then

$$v_k(T) - v_k(S) = \sum_{r=1}^{h} (v_k(S \cup \{i_1, i_2, \ldots, i_r\}) - v_k(S \cup \{i_1, i_2, \ldots, i_{r-1}\}))$$

$$= \sum_{r=1}^{h} M_{i_r}(S \cup \{i_1, i_2, \ldots, i_r\}, v_k)$$

$$\geq \sum_{r=1}^{h} M_{i_r}(T, v_k) = \sum_{i \in T \setminus S} M_i(T, v_k),$$

where the inequality follows from (3). So k-concavity implies that (N, v_k) is a total big boss game with k as big boss.

Suppose now that (N, v_k) is a total big boss game with k as big boss. First we prove that

$$M_i(U, v_k) \geq M_i(U \cup \{j\}, v_k) \tag{7}$$

for all $U \subset N$ and $i, j \in N \setminus \{k\}$ such that $\{i, k\} \subset U \subset N \setminus \{j\}$. By (6)

$$v_k(U \cup \{j\}) - v_k(U \setminus \{i\}) \geq M_j(U \cup \{j\}, v_k) + M_i(U \cup \{j\}, v_k). \tag{8}$$

On the other hand,

$$\begin{aligned} &v_k(U \cup \{j\}) - v_k(U \setminus \{i\}) \\ &\quad = v_k(U \cup \{j\}) - v_k(U) + v_k(U) - v_k(U \setminus \{i\}) \\ &\quad = M_j(U \cup \{j\}, v_k) + M_i(U, v_k). \end{aligned} \tag{9}$$

Then (7) follows directly from (8) and (9). To prove that (N, v_k) is k-concave take $S, T \subset N$ with $\{i, k\} \subset S \subset T$ and suppose that $T \setminus S = \{i_1, i_2, \ldots, i_h\}$. Apply (7) h times to obtain

$$M_i(S, v_k) \geq M_i(S \cup \{i_1\}, v_k) \geq M_i(S \cup \{i_1, i_2\}, v_k) \geq \cdots \geq M_i(T, v_k).$$

So, $M_i(S, v_k) \geq M_i(T, v_k)$ which implies (3). $\square$

From Muto et al. (1988) it follows that for a total big boss game with k as big boss we have for all T with $k \in T$ that the core $C(T, v)$ of the subgame (T, v) is equal to

$$C(T, v) = \left\{ x \in \mathbb{R}^T \mid 0 \leq x_i \leq M_i(T, v), i \in T \setminus \{k\}; \sum_{i \in T} x_i = v(T) \right\} \tag{10}$$

and the τ-value $\tau(T, v)$ (cf. Tijs (1981)) and the nucleolus $Nu(T, v)$ (cf. Schmeidler (1969)) coincide and are equal to the center z of the core $C(T, v)$ defined by

$$z_j = \begin{cases} M_j(T, v)/2, & j \in T \setminus \{k\}, \\ v(T) - \sum_{i \in T \setminus \{k\}} M_i(T, v)/2, & j = k. \end{cases} \tag{11}$$

These results about solution concepts inspired us to consider marginal-based allocation rules to compensate the informants. (See also Tijs, Timmer and Brânzei (2001).)

Take an IC game (N, v_k) and denote by P_k the set $\{S \subset N \mid k \in S\}$ of all coalitions containing the decision-maker. Call a scheme $[b_{i,S}]_{i \in S, S \in P_k}$ an allocation scheme if $[b_{i,S}]_{i \in S}$ corresponds to a core element of the subgame (S, v_k). Such an allocation scheme $[b_{i,S}]_{i \in S, S \in P_k}$ is called a *bi-monotonic allocation scheme* (bi-mas) if for all $S, T \in P_k$ with $S \subset T$ we have $b_{i,S} \geq b_{i,T}$ for all $i \in S \backslash \{k\}$, and $b_{k,S} \leq b_{k,T}$. Hence, in a bi-mas the big boss is better off in larger coalitions, and the other players are worse off.

Let the scheme $[b_{i,S}]_{i \in S, S \in P_k}$ be defined by $b_{k,S} = v_k(S)$ and $b_{i,S} = 0$ if $i \in S \backslash \{k\}$. Then $[b_{i,S}]_{i \in S, S \in P_k}$ is a bi-mas for (N, v_k). So, each IC game (N, v_k) has a bi-mas. The next theorem shows that a total big boss game has many bi-monotonic allocation schemes. This theorem is comparable to a result by Sprumont (1990) which tells that for a convex game each core element is extendable to a population monotonic allocation scheme. We say that a bi-mas $[b_{i,S}]_{i \in S, S \in P_k}$ is an extension of the core element $x \in C(N, v)$ if $x_i = b_{i,N}$ for all $i \in N$.

Theorem 5.2. *Let (N, v) be a total big boss game with k as big boss and let $x \in C(N, v)$. Then x is extendable to a bi-mas.*

Proof. Since $x \in C(N, v)$, (10) implies that we can find for each $i \in N \backslash \{k\}$ an $\alpha_i \in [0, 1]$, such that $x_i = \alpha_i M_i(N, v)$. Then

$$x_k = v(N) - \sum_{i \in N \backslash \{k\}} \alpha_i M_i(N, v).$$

We show that $[b_{i,S}]_{i \in S, S \in P_k}$, defined by $b_{i,S} = \alpha_i M_i(S, v)$ for all $i \in S \backslash \{k\}$ and $b_{k,S} = v(S) - \sum_{i \in S \backslash \{k\}} \alpha_i M_i(S, v)$, is a bi-mas. Then it is an extension of x.

Take $S, T \in P_k$ with $S \subset T$ and $i \in S \backslash \{k\}$. We have to prove that $b_{i,S} \geq b_{i,T}$ and $b_{k,S} \leq b_{k,T}$. First, $b_{i,S} = \alpha_i M_i(S, v) \geq \alpha_i M_i(T, v) = b_{i,T}$ because it follows from the proof of theorem 5.1 that (N, v) is k-concave. Second,

$$\begin{aligned}
b_{k,T} &= v(T) - \sum_{i \in T \backslash \{k\}} \alpha_i M_i(T, v) \\
&\geq \left(v(S) + \sum_{i \in T \backslash S} M_i(T, v) \right) - \sum_{i \in T \backslash \{k\}} \alpha_i M_i(T, v) \\
&= v(S) - \sum_{i \in S \backslash \{k\}} a_i M_i(T, v) + \sum_{i \in T \backslash S} (1 - \alpha_i) M_i(T, v) \\
&\geq v(S) - \sum_{i \in S \backslash \{k\}} \alpha_i M_i(S, v) + \sum_{i \in T \backslash S} (1 - \alpha_i) M_i(T, v) \\
&= b_{k,S} + \sum_{i \in T \backslash S} (1 - \alpha_i) M_i(T, v) \geq b_{k,S}
\end{aligned}$$

where the first inequality follows from (6), the second inequality from the k-concavity and the third inequality from the monotonicity of the total big boss game (N, v). So, $b_{k,T} \geq b_{k,S}$. $\square$

We end this Section with an example of a bi-mas.

Example 5.3. Consider again the IC game in example 2.2. This game (N, v'_4) is a total big boss game with player 4 as big boss. The bi-mas that assigns to each subgame (S, v'_4), $4 \in S$, the τ-value (see (11)) is given by the table below.

S	$\{4\}$	$\{1,4\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
$b_{1,S}$	–	30	–	–	20	20	–	0
$b_{2,S}$	–	–	30	–	20	–	20	0
$b_{3,S}$	–	–	–	30	–	20	20	0
$b_{4,S}$	60	90	90	90	120	120	120	160

Acknowledgements. Rodica Brânzei acknowledges financial support from the CentER at Tilburg University. Judith Timmer acknowledges financial support from the Netherlands Organization for scientific research (NWO) through project 613-304-059.

References

Brânzei R, Tijs S, Timmer J (2001) Collecting information to improve decision-making. International Game Theory Review 3:1–12

Muto S, Nakayama M, Potters J, Tijs S (1988) On big boss games. The Economic Studies Quarterly 39:303–321

Schmeidler D (1969) The nucleolus of a characteristic function game. SIAM Journal on Applied Mathematics 17:1163–1170

Shapley LS (1953) A value for n-person games. In: Kuhn HW, Tucker AW (eds.) Contributions to the theory of games II. Princeton University Press, Princeton, pp. 307–317

Shapley LS (1971) Cores of convex games. International Journal of Game Theory 1:11–26

Slikker M, Norde H, Tijs S (2000) Information sharing games. CentER discussion paper no. 2000-100, Tilburg University, The Netherlands

Sprumont Y (1990) Population monotonic allocation schemes for cooperative games with transferable utility. Games and Economic Behavior 2:378–394

Tijs SH (1981) Bounds for the core and the τ-value. In: Moeschlin O, Pallaschke D (eds.) Game theory and mathematical economics. North Holland, Amsterdam, pp. 123–132

Tijs SH (1990) Big boss games, clan games and information market games. In: Ichiishi T, Neyman A, Tauman Y (eds.) Game theory and applications. Academic Press, San Diego, pp. 410–412

Tijs SH, Timmer J, Brânzei R (2001) Compensations in information collecting situations. CentER discussion paper no. 2001-02, Tilburg University, The Netherlands